Classification of phase singularities
for complex scalar waves

Jiro ADACHI* and Go-o ISHIKAWA†

Abstract

Motivated by the importance and universal character of phase singularities which are clarified recently, we study the local structure of equi-phase loci near the dislocation locus of complex valued planar and spatial waves, from the viewpoint of singularity theory of differentiable mappings, initiated by H. Whitney and R. Thom. The classification of phase-singularities are reduced to the classification of planar curves by radial transformations due to the theory of A. du Plessis, T. Gaffney and L. Wilson. Then fold singularities are classified into hyperbolic and elliptic singularities. We show that the elliptic singularities are never realized by any Helmholtz waves, while the hyperbolic singularities are realized in fact. Moreover, the classification and realizability of Whitney's cusp, as well as its bifurcation problem are considered in order to explain the three points bifurcation of phase singularities. In this paper, we treat the dislocation of linear waves mainly, developing the basic and universal method, the method of jets and transversality, which is applicable also to non-linear waves.

1   Introduction.

A complex scalar wave has the locus, the dislocation locus, where its phase is not defined. The local structure of equi-phase loci near the dislocation locus is called a *phase-singularity* [21]. The phase singularities are called *optical vortices* in optics and are very basic and important objects in any science related to waves and quanta. In this paper we give the exhaustive classification of phase singularities of complex scalar waves of low codimension.

In [20][21], J. F. Nye constructed extensively complex scalar global planar waves satisfying the Helmholtz equation, with detailed analysis of those examples. Also he gave, by his examples, an explanation of an experimental bifurcation process of phase singularities: one degenerate singular point bifurcates

to three singular points and then another singular point annihilates with one of the three. He intends to explore the phase singularities from an analogy with the catastrophe theory [24][2].

In this paper, we understand phase singularities clearly using the singularity theory of differentiable mappings [3][19][25][7].

The planar complex scalar wave can be regarded, from the general point of view, simply as a differentiable mapping from the plane to the plane of complex numbers. Then, by a theorem of H. Whitney [26], the generic singularities of the wave, as a differentiable mapping, are just the *fold singularities* and the *cusp singularities*. The singular values form on the plane of complex numbers an immersed curve, the discriminant, with several number of cusps. Then generically the discriminant does not hit the zero, so that the zero is a regular value; generic phase singularities are regular, namely, locally diffeomorphic to the standard radial lines emitted from the origin. However, for a generic time-depending wave, the curve of singular values moves and momentarily may hit the zero. Thus, generically momentary wave can have degenerate phase singularities described by the fold singularities. Moreover, for a generic two parameter family of plane wave, the cusp singularity occurs as more complicated phase singularities.

The above simplified story must be examined twofold: First, in Whitney's theorem, the singularities are classified by means of arbitrary local diffeomorphisms of the source plane and the target plane. However, for the classification of phase singularities, we concern with the equi-phase lines and thus need to consider finer classification using only diffeomorphisms which preserve the radial lines on the target plane. Second, because waves must obey several natural conditions given by, say, the Helmholtz equations and the wave equations, more than just the differentiability, we must consider the realizability of singularities and determine generic singularities among waves satisfying those conditions.

We clarify the equivalence relations for phase singularities, and thus classify all phase singularities of low codimension, and discuss the realizability by the Helmholtz waves of phase singularities. Further, we propose the new explanation for the experimental bifurcation process treated in [20][21].

In the next section, we formulate our equivalence relation providing the base of our classification. A natural and refined classification by radial transformations is established on phase singularities for planar and spatial complex scalar waves. Then, we give the exact classification of generic complex planar waves without conditions motivated from physics.

In §3, the realization of singularities by the Helmholtz waves is examined by concrete examples, which have a different character with Nye's examples in [20].

In §4, we treat phase singularities of spatial complex scalar waves and consider their realizability.

In §5, the classification problem of planar waves is reduced to that of planar curves under diffeomorphisms preserving radial lines.

We introduce in §6 the notion of Helmholtz jet spaces and transversality to discuss genericity of singularities for Helmholtz waves.

2

In §7, as an application of the method developed in this paper, we discuss the bifurcation problem of phase singularities of solutions to non-linear Schrödinger equations.

In this paper, we consider local classification problem of phase singularities. For the global topology of dislocation locus, see [4][5].

For other applications of the singularity theory to solutions of partial differential equations, see [16][15][8] for instance.

2 Phase singularities for planar complex scalar waves.

We denote by $\mathbf{C}$ the plane of complex numbers and write a complex number as $u + \mathrm{i}w = re^{\mathrm{i}\theta}$, u, w being the real part and the imaginary part respectively, while r, θ the modulus (or the amplitude) and the argument (or the phase) respectively.

Let us consider a complex scalar wave

$$\Psi = \Psi(x, y, t) = u(x, y, t) + \mathrm{i}w(x, y, t)$$

on the (x, y)-plane depending on the time (or any other one-parameter). First we regard Ψ as just a time-depending complex valued function. We assume $u(x, y, t)$, $v(x, y, t)$ are differentiable (i.e. C^∞) functions.

If $\Psi(x, y, t) \neq 0$ at a point (x, y) and at a moment t, then we can write $\Psi(x, y, t) = r(x, y, t)e^{\mathrm{i}\theta(x, y, t)}$ uniquely with $r(x, y, t) > 0$ and $\theta(x, y, t)$ mod. 2π. Then, we are concerned with the wave dislocation locus at a moment $t = t_0$

$$\{(x, y) \mid \Psi(x, y, t_0) = 0\} = \{(x, y) \mid u(x, y, t_0) = 0, w(x, y, t_0) = 0\}$$

and the equi-phase curves $\{(x, y) \mid \theta(x, y, t_0) = \text{const.}\}$ outside of the wave dislocation locus.

Then, in the framework of singularity theory of differentiable mappings, we introduce the notion of *radial transformations* and give the exact classification results of singularities relatively to the radial transformations.

A *radial transformation* on $\mathbf{C}$ near 0 is a diffeomorphism, an invertible differentiable transformation, $\tau(u, w) = (U, W)$, $\tau \colon (\mathbf{C}, 0) \to (\mathbf{C}, 0)$ which sends any radial line $\{\theta = \text{const.}\}$ to a radial line. In fact, a diffeomorphism $\tau(u, w) = (U, W)$ is a radial transformation if and only if there exists a positive function $\rho(u, w)$ and real numbers a, b, c, d with $ad - bc \neq 0$ such that

$$U = \rho(u, w)(au + bw), \qquad W = \rho(u, w)(cu + dw).$$

For the classification, we define the equivalence relation on phase singularities: Two functions $\Psi(x, y, t)$ and $\Phi(x, y, t) = u'(x, y, t) + \mathrm{i}w'(x, y, t)$ are *radially equivalent* at points and moments (x_0, y_0, t_0) and (x_0', y_0', t_0') respectively, if there exist a local diffeomorphism $\sigma(x, y) = (X(x, y), Y(x, y))$ on the

plane with $X(x_0, y_0) = x_0', Y(x_0, y_0) = y_0'$ and a local radial transformation $\tau(u, w) = (U(u, w), W(u, w))$ near the origin on $\mathbf{C}$ such that

$$u(X(x, y), Y(x, y), t_0) = U(u'(x, y, t_0'), w'(x, y, t_0')),$$
$$w(X(x, y), Y(x, y), t_0) = W(u'(x, y, t_0'), w'(x, y, t_0')),$$

namely that $\Psi(\sigma(x, y), t_0) = \tau(\Phi(x, y, t_0'))$.

Theorem 2.1. *For a generic complex valued function* $\Psi(x, y, t)$*, the phase singularity at any point and any moment* (x_0, y_0, t_0) *is equivalent under radial transformations to the regular singularity*

$$\text{R:} \quad \psi(x, y) = x + \mathrm{i}y,$$

the hyperbolic singularity

$$\text{H:} \quad \psi(x, y) = x^2 - y^2 + \mathrm{i}y,$$

or to the elliptic singularity

$$\text{E:} \quad \psi(x, y) = x^2 + y^2 + \mathrm{i}y,$$

at the origin $(x, y) = (0, 0)$*. (see* Figure *1.)*

Each phase singularity of the classification in Theorem 2.1 is determined by its two jet actually.
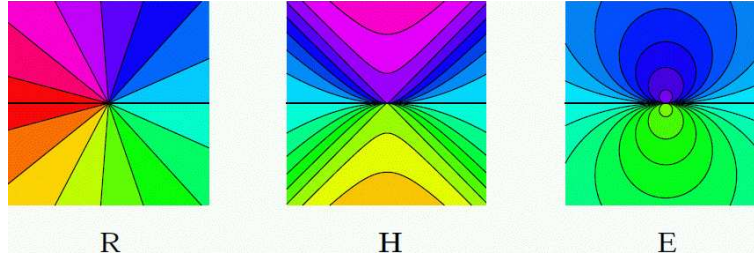

Figure 1: phase singularities

Remark 2.2. Besides the wave dislocation, we can classify generic critical points of phase functions defined outside of the dislocation locus: The generic critical points are the *non-degenerate maximal(minimal) points*, the *saddle points* and the *cuspidal points*. The last one bifurcates to one maximal(minimal) point and one saddle point.

Both hyperbolic and elliptic singularities are equivalent to the *fold singularity*

$$\psi : (x, y) \mapsto (u, w) = (x^2, y)$$

4

under arbitrary diffeomorphisms not necessarily radial transformation, namely, under the right-left equivalence.

For a momentary complex wave $\psi(x, y) = u(x, y) + iw(x, y)$ on the plane, the locus in $\mathbf{C}$ of complex values $\psi(x_0, y_0)$ for (x_0, y_0) with

$$
\det \left(\begin{array}{cc} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{array} \right) (x_0, y_0) = 0,
$$

is called the *discriminant* of the complex wave ψ.

Note that the above normal form of fold singularity is never generic as the phase singularity. In fact, the discriminant of ψ in that case is the w-axis in $\mathbf{C}$ which has the infinite tangency (actually coincides) with the radial lines $\{\theta = \pi/2\}$ and $\{\theta = 3\pi/2\}$. Generically the discriminant must be tangent to the radial lines in non-degenerate manner, namely, in the second order tangency. Then there are two possibility of non-degenerate tangency of the discriminant of fold singularities at $0 \in \mathbf{C}$; the image of ψ (the value set of ψ) is concave or convex. These correspond, respectively, to the hyperbolic singularity and elliptic singularity.

Moreover, the generic bifurcations on t of the hyperbolic singularities and the elliptic singularities are given by

$$
\begin{aligned}
\mathrm{H}_t : \quad \Psi(x, y, t) &= x^2 - y^2 + t + iy, \quad (t \in \mathbf{R}), \\
\mathrm{E}_t : \quad \Psi(x, y, t) &= x^2 + y^2 + t + iy, \quad (t \in \mathbf{R}).
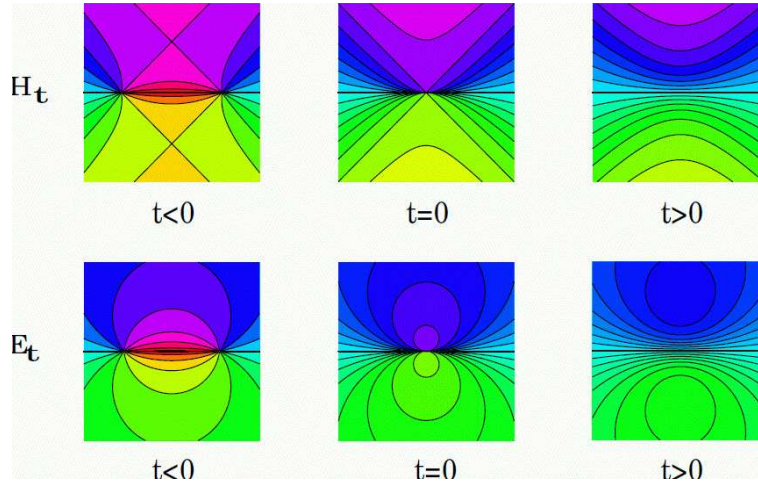\end{aligned}
$$

(see Figure 2.)

Figure 2: The bifurcations of the hyperbolic phase singularity (top) and the elliptic phase singularity (bottom).

The picture of the bifurcation of the hyperbolic singularity can be seen in Fig. 6 of [9].

Remark 2.3. The classification of phase singularities is closely related to the classification of plane curves under diffeomorphisms preserving a given singular foliation on the plane. Then, one of the most delicate cases is the case when the foliation is formed by radial lines, which is given by Euler vector field $X = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$ (cf. [27]). That is the case we are treating in this paper (see §5).

The discriminant of the fold singularity is a regular curve. Then degenerate phase singularities are classified as follows:

Proposition 2.4. *The phase singularities arising from fold singularities are classified into*

$$\psi_m(x, y) = x^2 \pm y^m + \mathrm{i}y, \quad m = 2, 3, 4, \ldots,$$

under radial transformations (and diffeomorphisms on the target), provided the discriminant curve has a contact with the tangent line at the origin in a finite multiplicity.

Remark 2.5. The generic bifurcation of $\psi_m(x, y)$ is described by the family

$$x^2 \pm y^m + t_{m-1}y^{m-1} + t_{m-2}y^{m-2} + \cdots + t_2 y^2 + t_0 + \mathrm{i}y,$$

with $(m-1)$-parameters $t_0, t_2, \ldots, t_{m-1}$.

A momentary complex wave $\psi \colon \left(\mathbf{R}^2, (x_0, y_0)\right) \to (\mathbf{C}, 0)$ is called a *Whitney's cusp* or simply a *cusp* if it is right-left equivalent (under local diffeomorphisms on $\mathbf{R}^2$ and $\mathbf{C}$ which are not necessarily radial) to the mapping $\psi(x, y) = x^3 + xy + \mathrm{i}y$. We are interested in this type of phase singularity because there occurs a three points bifurcation by just a translation $\psi_a(x, y) = x^3 + xy + \mathrm{i}(y + a), (a \in \mathbf{R})$. (For the classification of more degenerate singularities under the right-left equivalence relations, see [22][23]).

The Whitney's cusp appears generically in two parameter families of planar complex valued functions.

Then, we have

Proposition 2.6. (The radial classification of Whitney's cusps): *Any Whitney's cusp is equivalent under radial transformations to the standard function* $\psi(x, y) = x^3 + xy + \mathrm{i}y$.

The typical bifurcation of the phase singularities for a Whitney's cusp is described by $\psi_{a,b}(x, y) = x^3 + xy + b + \mathrm{i}(y + a), \ (a, b \in \mathbf{R})$. (see Figure 3)

Remark 2.7. The bifurcation problem of phase singularities arising from Whitney cusps is related to web geometry ([1], [10]). In fact, generic two parameter families of Whitney cusps define 3-webs on the plane, and their classification by radial transformations provides functional moduli (Remark 5.4).
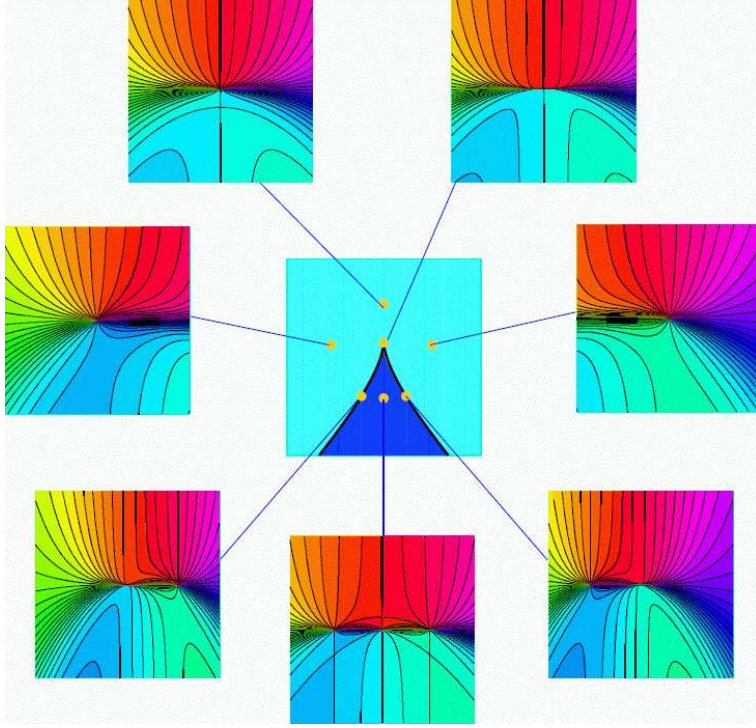
Figure 3: The two parameter bifurcation of the cusp phase singularity.

3   Phase singularities of the Helmholtz waves.

Now, we ask the physical reality of the classification; the instantaneous appearance of singularities for wave functions satisfying the wave equation and the Helmholtz equation. Namely, we assume the wave $\Psi(x, y, t)$ satisfies the wave equation

$$\frac{\partial^2 \Psi}{\partial t^2} = c^2 \nabla^2 \Psi,$$

for a positive real number c, where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the Laplacian. Moreover we assume that $\Psi(x, y, t)$ satisfies the Helmholtz equation

$$\nabla^2 \Psi + k^2 \Psi = 0,$$

for a positive real number k as a very natural physical assumption for monochromatic waves. We call a function which satisfies the Helmholtz equation the *Helmholtz function*. Further, we call the Helmholtz function with a parameter t which satisfies the wave equation as well the *Helmholtz wave*. Note that if we have a solution Ψ for $c = 1$, $k = 1$, then, by setting $\widetilde{\Psi}(x, y, t) = \Psi(kx, ky, \frac{c}{k}t)$, we have a solution $\widetilde{\Psi}$ for general c and k.

7

By solving the Cauchy problem properly, we obtain the following.

Proposition 3.1. *The complex valued function*

$$\psi(x, y) = \cos y - \cos x + \mathrm{i} \sin y$$

satisfies Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi = 0$ and has the hyperbolic singularity at the origin. Moreover the hyperbolic singularity with its generic bifurcation is realized by a Helmholtz wave

$$\Psi(x, y, t) = (\cos y - \cos x + \mathrm{i} \sin y) \cos t + \cos y \sin t,$$

(for $k = 1, c = 1$).

We see it is radially equivalent to the normal form simply by observing its Taylor expansion.

To the contrary, we observe:

Proposition 3.2. *Elliptic singularities are not realized as a function satisfying the Helmholtz equation.*

Proof: Suppose a function $\psi(x, y)$ is radially equivalent to the elliptic singularity. Then, the image of ψ is convex at the origin where the tangent line supports. Suppose the function $\psi(x, y)$ satisfies the Helmholtz equation with $k = 1$. From the equation $\psi_{xx} + \psi_{yy} + \psi = 0$, we see the Hessian of ψ is traceless at the dislocation locus $\{\psi = 0\}$, so are the real part $\mathrm{Re}(\mathrm{Hess}\psi)$ and the imaginary part $\mathrm{Im}(\mathrm{Hess}\psi)$. Thus, for any real numbers λ, μ,

$$\lambda \mathrm{Re}(\mathrm{Hess}\psi) + \mu \mathrm{Im}(\mathrm{Hess}\psi)$$

is never a definite matrix. However, the linear projection along the tangent line to the image of ψ must be definite. This leads to a contradiction. $\square$

In fact, as for the generic classification of phase singularities for one-parameter families of complex valued functions satisfying Helmholtz equation, we have:

Theorem 3.3. *The generic phase singularities of planar Helmholtz functions are regular singularities and hyperbolic singularities.*

Theorem 3.4. *The generic phase singularities of planar Helmholtz waves are regular singularities and hyperbolic singularities.*

Theorem 3.3, and Theorem 3.4 are proved in Section 6.

For the cusp singularities, we have:

Proposition 3.5. *A Whitney's cusp is realized as a Helmholtz wave. In fact,*

$$\psi(x, y) = x^3 \cos y + (x - 3xy) \sin y + \mathrm{i} \sin y$$

is a Whitney's cusp satisfying Helmholtz equation ($k = 1$): $\psi_{xx} + \psi_{yy} + \psi = 0$. Moreover,

$$\Psi(x, y, t) = (x^3 \cos y + (x - 3xy) \sin y + \mathrm{i} \sin y) \cos t + \mathrm{i} \cos y \sin t$$

gives a deformation of ψ by a Helmholtz wave ($k = 1, c = 1$) describing a three point bifurcation of the phase singularity.

Remark 3.6. By a similar construction to Proposition 3.5, we have another realization

$$\psi(x, y) = x^2 \cos y - y \sin y + i \sin y$$

of hyperbolic singularities.

Remark 3.7. Apart from the classification problem of phase singularities, we can show that generic Helmholtz function $\psi : \mathbf{R}^2 \to \mathbf{C}$ is, locally at any point in $\mathbf{R}^2$, *right-left equivalent* to a regular point, to a fold point or to a cusp point.

4 The radial classification of phase singularities for spatial complex scalar waves.

We study, in this section, the phase singularities of spatial waves $\Psi = \Psi(x, y, z, t) :$ $\mathbf{R}^3 \times \mathbf{R} \to \mathbf{C}$.

The generic singularities of differentiable mappings $\mathbf{R}^3 \to \mathbf{C}$ consist of the *definite fold singularities*, the *indefinite fold singularities* and the *cusp singularities*. The normal forms of them are given by

$$
\begin{aligned}
\text{the definite fold singularity}: \quad \psi(x, y, z) &= x^2 + y^2 + iz, \\
\text{the indefinite fold singularity}: \quad \psi(x, y, z) &= x^2 - y^2 + iz, \\
\text{the cusp singularity}: \quad \psi(x, y, z) &= x^3 + xy + z^2 + iy,
\end{aligned}
$$

under the left-right equivalence [14]. For a generic complex valued function $\Psi(x, y, z, t)$, only fold singularities may appear as a phase singularity. Moreover the discriminant curve has non-degenerate tangency with the tangent line at the origin. Thus we have

Theorem 4.1. *For a generic spatial complex valued function $\Psi(x, y, z, t)$, the phase singularity at any point and any moment (x_0, y_0, z_0, t_0) is equivalent, under the radial transformation, to the regular singularity*

$$\text{R:} \quad \psi(x, y, z) = x + iy,$$

to the definite hyperbolic singularity

$$\text{DH:} \quad \psi(x, y, z) = x^2 + y^2 - z^2 + iz,$$

to the definite elliptic singularity

$$\text{DE:} \quad \psi(x, y, z) = x^2 + y^2 + z^2 + iz,$$

or to the indefinite singularity

$$\text{I:} \quad \psi(x, y, z) = x^2 - y^2 - z^2 + iz,$$

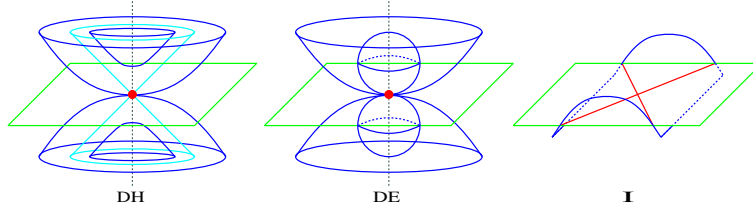*at the origin $(x, y, z) = (0, 0, 0)$. (See Figure 4.)*

Figure 4: Phase singularities for spatial waves

Similarly to the case of planar complex scalar waves, the generic bifurcations on t of the definite hyperbolic singularities, the definite elliptic singularities, and the indefinite singularities are given by

$$
\begin{aligned}
\text{DH}_t &: \Psi(x,y,z,t) = x^2 + y^2 - z^2 + t + \mathrm{i}z, \\
\text{DE}_t &: \Psi(x,y,z,t) = x^2 + y^2 + z^2 + t + \mathrm{i}z, \\
\text{I}_t &: \Psi(x,y,z,t) = x^2 - y^2 - z^2 + t + \mathrm{i}z.
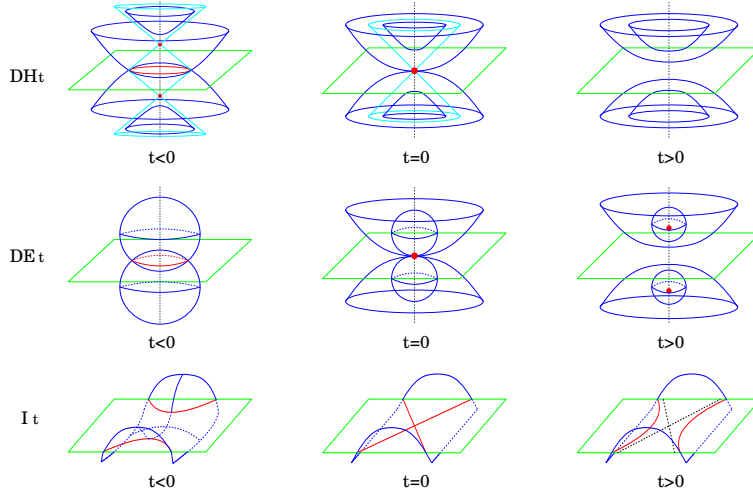\end{aligned}
$$

(see Figure 5.)

Figure 5: The bifurcations of phase singularities of spatial scalar waves.

For cusp singularities, we have

Proposition 4.2. *The phase singularities which come from cusp singularities are all radially equivalent to*

$$
\psi(x,y,z) = x^3 + xy + z^2 + \mathrm{i}y.
$$

10

Proposition 4.2 is induced from Lemma 5.3.

A complex valued function $\Psi(x, y, z, t)$ is called a *Helmholtz wave* if it satisfies the wave equation $\Psi_{tt} = c^2(\Psi_{xx} + \Psi_{yy} + \Psi_{zz})$ for a positive real number c and the Helmholtz equation $\Psi_{xx} + \Psi_{yy} + \Psi_{zz} + k^2\Psi = 0$ for a positive real number k.

As for the realizability of the spatial waves as Helmholtz waves, we have:

Proposition 4.3. *The definite hyperbolic singularity and the indefinite singularity together with their generic bifurcations are realized by Helmholtz waves (for $k = 1, c = 1$):*

$$\mathrm{DH_t} \quad : \quad \Psi(x, y, z, t) = (-\cos x - \cos y + 2\cos z + \mathrm{i}\sin z)\cos t + \cos z \sin t.$$

$$\mathrm{I_t} \quad : \quad \Psi(x, y, z, t) = (-2\cos x + \cos y + \cos z + \mathrm{i}\sin z)\cos t + \cos z \sin t.$$

Moreover, in a similar way to Proposition 3.2, we have

Proposition 4.4. *Definite elliptic singularities are not realized as Helmholtz waves.*

The cusp singularity is realized as a Helmholtz function:

Proposition 4.5. *The complex valued function*

$$\psi(x, y, z) = x^3\cos y + (x - 3xy)\sin y - \cos y + \cos z + \mathrm{i}\sin y$$

satisfies Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi_{zz} + \psi = 0$ and is radially equivalent to the cusp singularities.

5 Radial classification of planar curves.

The classification problem of complex waves under radial transformations is reduced to the classification problem of planar curves under radial transformations, by means of du Plessis, Gaffney, Wilson's theory [11][13]. The theory reduces the classification to that of discriminants with an exception. The exceptional cases are hyperbolic and elliptic singularities. Although they have the same discriminants, they are not radially equivalent. It depends on whether the image is convex or concave. The following results on curves have been applied to the classification of discriminants, and then, phase singularities.

Lemma 5.1. (The radial classification of regular curves): *A regular curve through the origin on $\mathbf{C}$ is transformed by radial transformations to the curve $u = w^m$ for some integer $m \geq 2$ or $u = g(w)$ for some function with null derivatives $g^{(i)}(0) = 0, i = 0, 1, 2, 3, \ldots$.*

Proof: Using a linear transformation, we may suppose the regular curve is given by $u = g(w)$ for a function $g(w)$. Suppose $\mathrm{ord}g = 0$ at $w = 0$ and $g^{(m)}(0) > 0$.

Then we can write $u = (a(w)w)^m$ for a function $a(w)$ with $a(0) > 0$. Set $\rho(w) = a(w)^{\frac{m}{m-1}}$, and define the radial transformation $U = \rho(w)u$, $W = \rho(w)w$. Then $U = \rho(w)\{a(w)W/\rho(w)\}^m = W^m$. $\qquad\square$

In Section 2, we study deformations of phase singularities. For them we observe the following.

Remark 5.2. Let $(u(t, \lambda), w(t, \lambda))$ be a deformation of the curve $(u(t, 0), w(t, 0)) = (t^m, t)$:

$$
\begin{aligned}
u(t, \lambda) &= \alpha_0(\lambda) + \alpha_1(\lambda)t + \cdots + \alpha_m(\lambda)t^m + \cdots, \\
w(t, \lambda) &= \beta_0(\lambda) + \beta_1(\lambda)t + \cdots,
\end{aligned}
$$

with $\alpha_0(0) = \cdots = \alpha_{m-1}(0) = \beta_0(0) = 0, \alpha_m(0) = \beta_1(0) = 1$. Then by a family of radial transformations, the family is transformed to

$$
\begin{aligned}
u(t, \lambda) &= \alpha_0'(\lambda) + \alpha_2'(\lambda)t^2 + \cdots + \alpha_{m-1}'(\lambda)t^{m-1} + t^m, \\
w(t, \lambda) &= \beta_0'(\lambda) + t,
\end{aligned}
$$

for some functions $\alpha_0'(\lambda), \alpha_2'(\lambda), \ldots, \alpha_{m-1}'(\lambda), \beta_0'(\lambda)$. The latter curve is expressed as

$$
u = t_0(\lambda) + t_1(\lambda)w + \cdots + t_{m-1}(\lambda)w^{m-1} + w^m.
$$

By a family of linear transformations $(u, w) \mapsto (u - t_1(\lambda)w, w)$, it is reduced to

$$
u = t_0(\lambda) + t_2(\lambda)w^2 + \cdots + t_{m-1}(\lambda)w^{m-1} + w^m.
$$

In general, any parametrized curve $(u(t), w(t))$ through the origin in $\mathbf{C}$ is equivalent by radial transformations and re-parametrizations to

$$
u(t) = t^m + O(t^{m+1}), \quad w(t) = t^n,
$$

for some integers m, n with $m > n$. We have $n = 1$ for regular curves. If $n \geq 2$, then we call the curve an (n, m)-*cusp*. A $(2, 3)$-cusp is called a *simple cusp* briefly a *cusp*.

Lemma 5.3. (The radial classification of simple cusps): *Any simple cusp is equivalent by radial transformations and re-parametrizations to*

$$
u(t) = t^3, \quad w(t) = t^2.
$$

Thus, any two simple cusps are radially equivalent to each other.

Lemma 5.3 implies Proposition 4.2 since the discriminant of $\psi(x, y, z) = x^3 + xy + z^2 + \mathrm{i}y$ is a simple cusp.

Proof of Lemma 5.3: Let $u(t) = t^3 + O(t^4), w(t) = t^2$ be a simple cusp. By a linear transformation on the (u, w)-plane and a re-parametrization of t, we may

12

suppose the curve is given by $u(t) = t^3$, $w(t) = t^2 + O(t^4)$. Set $w(t) = t^2 a(t)$ for a smooth function $a(t)$. Then $a(0) = 1, a'(0) = 0$. Then there exists a smooth function $\rho(x, y)$ such that $a(t) = \rho(t^2, t^3)$ by the preparation theorem ([14]). Then the curve is radially equivalent to the curve

$$u(t) = \frac{1}{\rho(t^2, t^3)^3} t^3, \quad w(t) = \frac{1}{\rho(t^2, t^3)^2} t^2,$$

which is radially equivalent to $u(t) = t^2, w(t) = t^3$. $\qquad\qquad\square$

Remark 5.4. Let $C_{a,b}(t) = (u(t, a, b), w(t, a, b))$ be a generic two parameter family of simple cusps. For each (a, b), we draw tangent lines to the simple cusp $C_{a,b}$ from the origin. Then there exists a non-void open subset U such that for $(a, b) \in U$, there are exactly three tangent rays. By the assignment of the corresponding tangent points, we have three functions λ_1, λ_2, λ_3 on U; $C_{a,b}(\lambda_1)$, $C_{a,b}(\lambda_2), C_{a,b}(\lambda_3)$ are tangent points. Thus we have a triple of foliations:

$$\lambda_1 = \text{const.}, \ \lambda_2 = \text{const.}, \ \lambda_3 = \text{const.},$$

that is, a 3-web on U. Moreover, radially equivalent families of simple cusps have isomorphic 3-webs. It is known that the classification of 3-webs has function moduli in general ([10]).

6 Helmholtz jet space and transversality.

We introduce the notion of Helmholtz jet spaces and show the transversality theorem in a Helmholtz jet space, as one of the main ideas to show the results in this paper. Note that, in [15], analogous jet spaces are considered for other kinds of Monge-Ampère equations.

Consider the Taylor expansion around $(x, y) = (x_0, y_0)$ of a complex valued function ψ on the (x, y)-plane:

$$\psi(x, y) = a + bX + cY + \frac{e}{2}X^2 + fXY + \frac{g}{2}Y^2 + \frac{h}{6}X^3 + \frac{k}{2}X^2Y + \frac{\ell}{2}XY^2 + \frac{m}{6}Y^3 + \cdots .$$

Here we set $X = x - x_0$, $Y = y - y_0$, and $a, b, c, \ldots$ are complex numbers.

Suppose ψ is a Helmholtz function for $k = 1$, that is, ψ satisfies Helmholtz equation $\psi_{xx} + \psi_{yy} + \psi = 0$. Then we have

$$e + g + a = 0, \ h + \ell + b = 0, \ k + m + c = 0.$$

Therefore, we have

$$\begin{aligned}
\psi(x, y) &= a + bX + cY + \frac{e}{2}X^2 + fXY - \frac{1}{2}(a + e)Y^2 \\
&\quad + \frac{h}{6}X^3 + \frac{k}{2}X^2Y - \frac{1}{2}(b + h)XY^2 - \frac{1}{6}(c + k)Y^3 + \cdots .
\end{aligned}$$

The Taylor expansion of a function ψ up to order r around a point (x_0, y_0) of $\mathbf{R}^2$ is called the *r-jet* of ψ at (x_0, y_0) and denoted by $j^r\psi(x_0, y_0)$. Denote

13

by $J^r(\mathbf{R}^2, \mathbf{C})$ the space of r-jets of complex valued functions on $\mathbf{R}^2$. In it, we denote by $J^r_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$ the set of r-jets of planar Helmholtz functions for $k = 1$:

$$J^r_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C}) = \{j^r \psi(x_0, y_0) \mid \psi_{xx} + \psi_{yy} + \psi = 0 \text{ around } (x_0, y_0)\}.$$

We call it the *Helmholtz r-jet space*. For example, $J^3_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$ is identified with $\mathbf{R}^{16} = \mathbf{R}^2 \times \mathbf{C} \times \mathbf{C}^6$ with coordinates x_0, y_0; $a = a_1 + ia_2$; $b = b_1 + ib_2$, $c = c_1 + ic_2$, $e = e_1 + ie_2$, $f = f_1 + if_2$, $h = h_1 + ih_2$, and $k = k_1 + ik_2$.

In general, the Taylor expansion of a Helmholtz function $\psi(x, y)$ defined around (x_0, y_0) is determined by $\psi(x, y_0)$ and $\psi_y(x, y_0)$. Moreover, for any given complex valued analytic functions $\psi_0(x)$ and $\psi_1(x)$ defined around x_0, there exists uniquely a complex valued function $\psi(x, y)$ defined around (x_0, y_0) satisfying the Helmholtz equation, $\psi(x, y_0) = \psi_0(x)$ and $\psi_y(x, y_0) = \psi_1(x)$. The r-jet of ψ at (x_0, y_0) is determined by the r-jet of $\psi_0(x)$ at x_0 and $(r-1)$-jet of $\psi_1(x)$ at x_0. Thus, $J^r_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$ is identified with $\mathbf{R}^N$ for some natural number N. With any Helmholtz function ψ defined around (x_0, y_0), there is associated a mapping

$$j^r \psi \colon (\mathbf{R}^2, (x_0, y_0)) \to J^r_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$$

defined by taking the r-jet of ψ at (x, y) for each (x, y) near (x_0, y_0). It is called the *r-jet extension* of ψ. Moreover, to any family $\Psi(x, y, \lambda) \colon \mathbf{R}^2 \times \mathbf{R}^\ell \to \mathbf{C}$ of Helmholtz functions, there corresponds a mapping

$$j^r \Psi \colon (\mathbf{R}^2 \times \mathbf{R}^\ell, (x_0, y_0, \lambda_0)) \to J^r_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$$

by taking the r-jet of $\Psi(x, y, \lambda')$ at (x, y) for each (x, y) near (x_0, y_0) and parameter λ' near λ_0.

By a similar proof to that of ordinary transversality theorem([14]), we have:

Lemma 6.1. *Suppose a finite number of submanifolds $W_1, W_2, \ldots$ of Helmholtz r-jet space $J^r_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$ are given. Then, any Helmholtz function $\psi(x, y)$ defined around (x_0, y_0) is approximated (in C^∞ topology) by a Helmholtz function $\tilde{\psi}(x, y)$ defined around (x_0, y_0) such that the r-jet extension of $\tilde{\psi}(x, y)$ is transversal to any W_i. Moreover, any Helmholtz wave $\Psi(x, y, t)$ is approximated by a Helmholtz wave $\widetilde{\Psi}(x, y, t)$ such that $j^r \Psi$ is transversal to any W_i.*

By using Lemma 6.1, we show Theorem 3.3 and Theorem 3.4. In $J^3_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$ with coordinates x_0, y_0; $a = a_1 + ia_2$; $b = b_1 + ib_2$, $c = c_1 + ic_2$, $e = e_1 + ie_2$, $f = f_1 + if_2$, $h = h_1 + ih_2$ and $k = k_1 + ik_2$, we set

$$W_1^6 = \{a = 0, \ b = 0, \ c = 0\},$$

which is of codimension 6. Moreover, we define a submanifold W_3^4 of codimension 4 by the equations

$$\left| \begin{array}{c} \begin{vmatrix} e_1 & c_1 \\ e_2 & c_2 \end{vmatrix} + \begin{vmatrix} b_1 & f_1 \\ b_2 & f_2 \end{vmatrix} \\ b_1 \end{array} \quad \begin{array}{c} \begin{vmatrix} f_1 & c_1 \\ f_2 & c_2 \end{vmatrix} - \begin{vmatrix} b_1 & a_1 + e_1 \\ b_2 & a_2 + e_2 \end{vmatrix} \\ c_1 \end{array} \right| = 0,$$

14

and

$$
\left| \begin{array}{cc}
\left| \begin{array}{cc} e_1 & c_1 \\ e_2 & c_2 \end{array} \right| + \left| \begin{array}{cc} b_1 & f_1 \\ b_2 & f_2 \end{array} \right| & \left| \begin{array}{cc} f_1 & c_1 \\ f_2 & c_2 \end{array} \right| - \left| \begin{array}{cc} b_1 & a_1+e_1 \\ b_2 & a_2+e_2 \end{array} \right| \\[6pt]
b_2 & c_2
\end{array} \right| = 0,
$$

together with $a = 0$, $b_1 c_2 - b_2 c_1 = 0$, minus a locus W_2^5 of more degenerate singularities, which is of codimension ≥ 5. The definition of W_3^4 is from the idea of the iterated Jacobian [12]. Further, we set

$$W_4^3 = \{a = 0,\ b_1 c_2 - b_2 c_1 = 0\} \setminus (W_1^6 \cup W_2^5 \cup W_3^4),$$

which is of codimension 3, and

$$W_5^2 = \{a = 0\} \setminus (W_1^6 \cup W_2^5 \cup W_3^4 \cup W_4^3),$$

which is of codimension 2.

From Lemma 6.1, any Helmholtz function $\psi(x, y)$ is approximated to a Helmholtz function whose r-jet extension is transversal to the above submanifolds in $J^3_{\mathrm{Helm}}(\mathbf{R}^2, \mathbf{C})$. This implies the following from the Whitney theory ([14]). The transversality of $j^3\psi$ at (x_0, y_0) to W_5^2 implies that ψ has the regular phase singularity at (x_0, y_0). Similarly, the transversality to W_4^3 implies the fold singularity, and W_3^4 the cusp singularity as a mapping from a plane to a plane. When a function has fold singularity, generically, there are two possibilities of phase singularities: hyperbolic and elliptic singularities. However, from Proposition 3.2, there is no elliptic phase singularity for Helmholtz functions. This shows Theorem 3.3. Furthermore, Lemma 6.1 claims that the transversality theorem holds even for Helmholtz waves. Therefore, Theorem 3.4 is proved in the same way as above.

Remark 6.2. The transversality to W_2^5 and W_1^6 means that $j^3\psi$ does not intersect to W_2^5 and W_1^6, for the two parameter family of Helmholtz functions.

7 Bifurcation problem of phase singularities of non-linear Schrödinger waves.

We can apply our method to study phase singularities appearing in non-linear waves. Actually we treat local analytic waves or formal waves. Note that we would have to find other methods for the study of global structure of phase singularities of non-linear waves due to the existence of soliton solutions (see for instance [17]),

Let

$$i\Psi_t + \frac{1}{2}\Psi_{xx} + f(x, \Psi) = 0$$

be a Schrödinger equation for a complex valued function $\Psi = \Psi(t, x)$. Here f is a real analytic function on $\mathbf{R} \times \mathbf{C}$, for instance, $f(x, \Psi) = |\Psi|^2 \Psi$.

Set $\Psi = u + \mathrm{i}w$. Then, in the case $f(x, \Psi) = |\Psi|^2\Psi$ the equation reads

$$\begin{cases} u_{xx} & = & 2w_t - 2(u^2 + w^2)u \\ w_{xx} & = & -2u_t - 2(u^2 + w^2)w \end{cases}$$

Let us denote by

$$J_S^r(\mathbf{R}^2, \mathbf{C}) := \{j^r\Psi(t_0, x_0) \mid \mathrm{i}\Psi_t + \frac{1}{2}\Psi_{xx} + f(x, \Psi) = 0\}$$

the *Schrödinger jet space*. Then we see $J_S^r(\mathbf{R}^2, \mathbf{C})$ is a submanifold of $J^r(\mathbf{R}^2, \mathbf{C})$.

In $J_S^r(\mathbf{R}^2, \mathbf{C})$, the condition $\Psi = 0$ gives a smooth submanifold of $J_S^r(\mathbf{R}^2, \mathbf{C})$ of codimension 2. Therefore, the phase singularities appear on the (t, x)-plane at isolated points and, on the x-line, the phase singularities appear momentarily. The fold singularities form a submanifold of codimension 3 in $J_S^2(\mathbf{R}^2, \mathbf{C})$. The fold locus is tangent to the x-line in codimension 4.

The condition $\Psi = \Psi_x = 0$ also gives a smooth submanifold of $J_S^r(\mathbf{R}^2, \mathbf{C})$ of codimension 4. Therefore, degenerate phase singularities appear in a generic two parameter family of solutions, where $\Psi = \Psi_x = 0$. Consider the condition $\Psi = \Psi_x = \Psi_{xx} = 0$. If $\Psi(t_0, x_0) = \Psi_x(t_0, u_0) = 0$, then the condition $\Psi_{xx}(t_0, x_0) = 0$ is equivalent to that $\mathrm{i}\Psi_t(t_0, x_0) + f(x_0, 0) = 0$. If $f(x_0, 0) = 0$, for instance if $f(x, \Psi) = |\Psi|^2\Psi$, then $\Psi_{xx}(t_0, x_0) = 0$ if and only if $\Psi_t(t_0, x_0) = 0$. Thus any bifurcation on t of phase singularity with $\Psi(t_0, x_0) = \Psi_x(t_0, x_0) = \Psi_{xx}(t_0, x_0) = 0$ is degenerate ($\Psi_t(t_0, x_0) = 0$).

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Jiro ADACHI
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
E-mail : j-adachi@math.sci.hokudai.ac.jp

Go-o ISHIKAWA
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan.
E-mail : ishikawa@math.sci.hokudai.ac.jp
E-mail : ishikawa@topology.coe.hokudai.ac.jp